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Correlation Functions and the Goldstone Picture for the Hierarchical Classical Vector Model at Low Temperatures in Three or More Dimensions

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Low-temperature properties of the one-and two-point correlation functions are obtained for the pure state classical vector model in a hierarchical formulation. We consider the Z^d lattice model ($d \geqslant 3$) where the single-site spin variable $\phi \in R^\nu$ has a density proportional to $e^{-\lambda(\phi^2-1)^2}$ for large $\lambda \leqslant \infty$. We obtain the pure state one- and two-point functions by introducing a uniform magnetic field which goes to zero as the volume goes to infinity. Using renormalization group methods, we generate a sequence of effective actions and spin variable and determine the spontaneous magnetization (one-point function parallel to the field). We confirm the Goldstone picture by showing that the truncated two-point function has the canonical massless decay $|x-y|^{-(d-2)}$, $x, y \in Z^d$ in the directions perpendicular to the field. We show a faster decay in the parallel direction and for large d that the decay is $|x-y|^{-(d+2)}$.

KEY WORDS: Hierarchical classical vector model; Goldstone picture; infrared asymptotic freedom; renormalization group method; correlation functions.

1. INTRODUCTION AND RESULTS

In a previous paper,[1] hereafter referred to as I, we obtained low-temperature thermodynamic properties of a hierarchical version of the Z^d, $d \geqslant 3$, lattice classical vector spin model with partition given by

$$Z = \int \exp\{\beta[\tfrac{1}{2}(\phi, \Delta\phi) + (h, \phi_1)]\} \prod_x \delta(|\phi(x)|^2 - 1)\, d\phi(x) \qquad (1.1)$$

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where $\phi = (\phi_1, \phi_2, ..., \phi_\nu) \in R^\nu$, h is a uniform magnetic field in the one-direction, Δ is the lattice Laplacian, and β is the inverse temperature. See refs. 2–7 for results on the model of (1.1). In I, renormalization group methods (see also refs. 8 and 9) were used to obtain an expression for the free energy and spontaneous magnetization, calculated as the thermodynamic limit of the derivative of the free energy per site with respect to the magnetic field at zero field. A sequence of magnetic fields, going to zero as the volume goes to infinity, is used to put the system in a pure state. In this paper we obtain low-temperature properties of the pure state one- and two-point correlation functions at zero magnetic field. The result for the one-point function follows immediately from I, using its translational invariance. In agreement with the Goldstone picture,[2,3] i.e., there are $\nu - 1$ massless degrees of freedom, we show that the truncated two-point function perpendicular to the field decays as $|x-y|^{-(d-2)}$, $x, y \in Z^d$; the parallel one decays faster and for large d the decay is $|x-y|^{-(d+2)}$. See refs. 10–12 for results for other hierarchical models.

Specifically, the model we consider is obtained from the above by replacing Δ by the hierarchical Laplacian and relaxing the fixed spin condition, i.e., the partition function on the lattice $\Lambda_N = [-L^N/2, L^N/2]^d \subset Z^d$, L odd, is given by (after the change of variables $\phi \to \beta^{-1/2}\phi$)

$$Z_N(h, \beta) = \int \exp\left\{\beta^{1/2} h \sum_{x \in \Lambda_N} \phi_1(x) - \frac{\lambda}{\beta} \sum_{x \in \Lambda_N} [\phi(x)^2 - \beta]^2\right\} d\mu_N(\phi) \tag{1.2}$$

where $\beta < \infty$ and $\lambda \leqslant \infty$ are taken to be large, and $d\mu_N(\cdot)$ is a Gaussian probability measure with covariance G_N given by the inverse of the hierarchical Laplacian. We introduce the Gaussian probability measure $d\mu_m$ with covariance G_m for functions on $\Lambda_m = [-L^m/2, L^m/2]^d \subset Z^d$; for $m = 0$ only a single site is present and $G_0 = (1 - L^{-(d-2)})^{-1}$. The term G_m is given by

$$G_m(x, y) = (1 - L^{2-d})^{-1} L^{(2-d)[n(x,y)-1]} \tag{1.3}$$

for all $x, y \in \Lambda_m$ and $n(x, y) = \min\{n = \{1, 2, ...\}: [L^{-n}x] = [L^{-n}y]\}$, where, for any $u \in R^d$, $[u]$ is the element of Z^d such that $-1/2 \leqslant u_i - [u]_i < 1/2$. We remark that the measure $d\mu_m$ is determined by the relation

$$\int \exp[i(\phi, J)]\, d\mu_m(\phi) = \exp[-\tfrac{1}{2}(J, G_m J)]$$

via Bochner's theorem. For $m \geqslant 1$, $d\mu_m(\phi)$ does not have a density as a function, but as a distribution, since G_m has a nonzero null space. Furthermore, using the spectral representation of G_m (which can be obtained explicitly) and restricting G_m and the measure to the orthogonal complement of the null space, one can write the Gaussian measure with a density. The associated quadratic form is long range. This procedure provides a link between the nonlocal part of the interaction in our model and the one studied in ref. 11.

The G_m satisfy the recursion relation

$$G_{n+1}(Lx+u, Ly+v) = L^{2-d}G_n(x, y) + \delta_n(x, y)$$

for all $x, y \in \Lambda_n$ and u, v such that $-L/2 < u_\alpha, v_\alpha < L/2$; $\delta_n(x, y)$ is the Kronecker δ.

We use the fundamental relation

$$\int \prod_{x \in \Lambda_N} f_x(\phi(x))\, d\mu_n = \int \prod_{x \in \Lambda_{N-1}} f'_x(\phi(x))\, d\mu_{n-1} \tag{1.4}$$

where

$$f'_x(\phi) = \int \prod_{u \in B_0^{(1)}} f_{Lx+u}(L^{-(d-2)/2}\phi + \xi)\, e^{-\xi^2/2}\, d\xi$$

with $B_0^{(1)} = \{u \in Z^d;\ -L/2 < u_i < L/2\}$. The relation (1.4) is derived by differentiating the generating function $\exp[\frac{1}{2}(J, G_n J)]$ with respect to the J's using the decompositions $\phi(Lx+u) = L^{-1/2(d-2)}\phi'(x) + \eta(x)$, $x \in \Lambda_{n-1}$, $d\mu_n(\phi) = d\mu_{n-1}(\phi')\, d\mu_{n-1}(\eta)$, where $d\mu_n(\eta) = \prod_{x \in \Lambda_n} d\mu(\eta(x))$ and $d\mu(\eta(x))$ is a Gaussian probability measure with covariance 1. We adopt the convention that the ξ integral includes the factor $(2\pi)^{-\nu/2}$. We can look at Eq. (1.4) as giving a decomposition of an original field integral into a new field and fluctuation field integral (see I for more details). We define the renormalization group transformation (RGT) R by

$$e^{-RW(\phi)} = \int \exp[-L^d W(L^{-(d-2)/2}\phi + \xi)]\, e^{-\xi^2/2}\, d\xi \tag{1.5}$$

A simple calculation shows that the derivative of R is given by the linear operator

$$\begin{aligned}\frac{dR(W)}{dW} g(\phi) &= \frac{L^d \int g(L^{-(d-2)/2}\phi + \xi) \exp[-L^d W(L^{(d-2)/2}\phi + \xi)]\, e^{-\xi^2/2}\, d\xi}{\int \exp[-L^d W(L^{-(d-2)/2}\phi + \xi)]\, e^{-1/2\xi^2}\, d\xi} \\ &\equiv L^d M_W g(\phi)\end{aligned} \tag{1.6}$$

so that by the chain rule

$$\frac{dR^n(W)}{dW} g(\phi) = L^{nd} M_{R^{n-1}W} \cdots M_W g(\phi) \tag{1.7}$$

Applying Eq. (1.4) to the partition function Z_N and one-point function gives, letting $V=(\lambda/\beta)(\phi^2-\beta)^2$, $V_h=V-\beta^{1/2}hI$, and $I(\phi)=\phi_1$,

$$\begin{aligned} Z_N &= \int \prod_{x\in\Lambda_N} \exp[-V_h(\phi(x))]\, d\mu_N \\ &= \cdots = \int \prod_{x\in\Lambda_1} \exp[-R^{N-1}V_h(\phi(x))\, d\mu_1 \\ &= \int \exp[-R^N V_h(\phi)]\, d\mu_0 \end{aligned} \tag{1.8}$$

$$\begin{aligned} \langle\phi_i(0)\rangle^{(N)} &\equiv Z_N^{-1}\beta^{-1/2}\int \phi_i(0) \prod_{x\in\Lambda_N} \exp[-V_h(\phi(x))]\, d\mu_N \\ &= Z_N^{-1}\beta^{-1/2}\int M_{V_h}(\phi_i(0)) \prod_{x\in\Lambda_{N-1}} \exp[-RV_h(\phi(x))]\, d\mu_{N-1} \\ &= \cdots = Z_N^{-1}\beta^{-1/2}\int M_{R^{N-1}V_h}\cdots M_{V_h}(\phi_i)\exp[-R^N V_h(\phi)]\, d\mu_0 \end{aligned} \tag{1.9}$$

For the two-point function let $x=L^n z+L^{n-1}u_1+\cdots+u_n$ and $y=L^n z+L^{n-1}v_1+\cdots+v_n$, $z\in\Lambda_n$, $u_1\neq v_1$, where $n=n(x,y)\geqslant 1$ defines the hierarchy containing the points x and y, u_i, $v_j\in B_0^{(1)}$. We consider scaled correlation functions due to the change of variables in the original partition function of Eq. (1.1). Using Eq. (1.4) in the two-point function gives

$$\begin{aligned} &\langle\phi_i(x)\,\phi_i(y)\rangle^{(N)} \\ &\equiv Z_N^{-1}\beta^{-1}\int \phi_i(x)\,\phi_i(y) \prod_{w\in\Lambda_N} \exp[-V(\phi(w))]\, d\mu_N \\ &= Z_N^{-1}\beta^{-1}\int M_{V_h}[\phi_i(L^{n-1}z+L^{n-2}u_1+\cdots+u_{n-1})] \\ &\quad\times M_{V_h}[\phi_i(L^{n-1}z+L^{n-2}v_1+\cdots+v_{n-1})] \\ &\quad\times \prod_{w\in\Lambda_{N-1}} \exp[-RV_h(\phi(w))]\, d\mu_{N-1} \\ &= \cdots = Z_N^{-1}\beta^{-1}\int M_{R^{n-1}V_h}[M_{R^{n-2}V_h}\cdots M_{V_h}(\phi_i(z))]^2 \\ &\quad\times \prod_{w\in\Lambda_N} \exp[-R^n V_h(\phi(w))\, d\mu_n \\ &= \cdots = Z_N^{-1}\beta^{-1}\int M_{R^{N-1}V_h}\cdots \\ &\quad\times M_{R^{n-1}V_h}[M_{R^{n-2}V_h}\cdots M_{V_h}(\phi_i)]^2\exp[-R^N V_h(\phi)]\, d\mu_0 \end{aligned} \tag{1.10}$$

From Eqs. (1.9) and (1.10) we see that we need to control $R^k V_h$ and compositions of M operators with a magnetic field. In I we showed that there is a simple relation between $R^n V_h$ and $R^n V$, namely, the linear shift formula

$$R^n(V-\rho I_k) = T_{-a'_n\rho e_k} R^n V - L^{n/2(d+2)}\rho I_k - \tfrac{1}{2}L^{(n+1)d}(L^{2n}-1)(L^2-1)^{-1}\rho^2 \tag{1.11}$$

where

$$a'_n = L^d L^{(n/2)(d-2)}\left(\frac{L^{2n}-1}{L^2-1}\right), \qquad I_k(\phi)=\phi_k$$

e_k is the unit vector in the k direction and T_{-b}, $b\in R^\nu$, the translation operator $T_{-b}\,g(\phi)=g(\phi+b)$. Using Eqs. (1.7) and (1.11), we obtain, letting $e_1=e$ and $a_n=L^{-nd}a'_n$,

$$\begin{aligned} M_{R^{n-1}V_h}\cdots M_{V_h}f(\phi) &= L^{-nd}\frac{d}{d\varepsilon}R^n(V_h+\varepsilon f)\Big|_{\varepsilon=0}(\phi)\\ &= L^{-nd}\frac{d}{d\varepsilon}T_{-a'_n\beta^{1/2}he}R^n(V+\varepsilon f)\Big|_{\varepsilon=0}\\ &= T_{-a'_n\beta^{1/2}he}M_{R^{n-1}V}\cdots M_V f(\phi) \end{aligned} \tag{1.12}$$

and

$$\begin{aligned} M_{R^{n-1}V}\cdots M_V I_k(\phi) &= L^{-nd}\frac{d}{d\varepsilon}R^n(V+\varepsilon I_k)\Big|_{\varepsilon=0}(\phi)\\ &= L^{-(n/2)(d-2)}\phi_k - a_n\partial_k R^n V(\phi) \end{aligned} \tag{1.13}$$

We use Eqs. (1.12) and (1.13) in Eqs. (1.9) and (1.10) to obtain, after making the change of variables $\phi\to\phi+a'_N\beta^{1/2}he$,

$$\begin{aligned} \langle\phi_k(0)\rangle^{(N)} = \beta^{-1/2}&\left\{\int\left[L^{-(N/2)(d-2)}\phi_k - a_N\frac{\partial R^N V}{\partial\phi_k}(\phi)\right]\right.\\ &\left.\times\exp[-U_N(\phi,\beta,h)]\,d\phi\right\}\\ &\times\left\{\int\exp[-U_N(\phi,\beta,h)]\,d\phi\right\}^{-1} \end{aligned} \tag{1.14}$$

$$\begin{aligned}
&\langle \phi_i(x)\,\phi_i(y)\rangle^{(N)}\\
&\quad=\beta^{-1}L^{-(N-n)d}\int\left(\frac{dR^{N-n}}{dW}(R^nV)\right.\\
&\qquad\times\left\{\left[L^{-[(n-1)/2](d-2)}\phi_i-a_{n-1}\frac{\partial R^{n-1}V(\phi)}{\partial\phi_i}\right]\right.\\
&\qquad\times\left.\left.\left[L^{-[(n-1)/2](d-2)}\phi_i-a_{n-1}\frac{\partial R^{n-1}V(\phi)}{\partial\phi_i}\right]\right\}\right)\\
&\qquad\times\exp[-U_N(\phi,\beta,h)]\,d\phi\bigg/\int\exp[-U_N(\phi,\beta,h)]\,d\phi
\end{aligned}\tag{1.15}$$

where, setting $r_N\equiv(L^d-1)(L^2-1)^{-1}-L^{-2N}(L^d-L^2)(L^2-1)^{-1}$,

$$U_N(\phi,\beta,h)=R^NV(\phi)-r_NL^{(N/2)(d+2)}\beta^{1/2}h\phi_1+\tfrac{1}{2}(1-L^{2-d})\,\phi^2 \tag{1.16}$$

Through the use of Eqs. (1.12) and (1.13), we see that we can consider effective spin variables with zero magnetic field, but with a linearly shifted effective action. From the results of I, recall that

$$R^NV=d_N+4\lambda_N(|\phi|-\beta_N^{1/2})^2+w_N(|\phi|-\beta_N^{1/2})\equiv d_N+V^{(N)}$$

for $||\phi|-\beta_N^{1/2}|<\beta_N^\alpha$, $0<\alpha$ and small, where $w_N(\beta)$ is analytic and $|w_N(\sigma)|<k\beta_N^{3\alpha-1/2}$. The λ_N converges to λ^*, the fixed point of the function $f(\lambda)=L^2\lambda/(1+8L^d\lambda)$, i.e., to $\lambda^*=(L^2-1)/8L^d$. Later we specify the rate of convergence more precisely. Also we see that the effective spin variable is modified from its canonical value $L^{-(n/2)(d-2)}\phi$ by a term proportional to the derivative of the nth effective action.

We now consider the one-point function. In (1.14) an integration by parts of the second term gives

$$\begin{aligned}
\langle\phi_k(0)\rangle^{(N)}&=-hL^{2N+d}(1-L^{-2N})(L^2-1)^{-1}r_N\delta_{1k}+\beta^{-1/2}r_N\\
&\quad\times L^{-(N/2)(d-2)}\frac{\int\phi_k\exp[-U_N(\phi)]\,d\phi}{\int\exp[-U_N(\phi)]\,d\phi}\\
&=\frac{\partial}{\partial h}[\beta^{-1}L^{-Nd}\ln Z_N(h)]
\end{aligned}\tag{1.17}$$

which is precisely the finite-volume magnetization per unit site calculated in I. Actually, this follows from translation invariance of the finite-volume one-point function even though the covariance G_N is not translation invariant. Thus, using the results of I, we have, letting, as in I, the magnetic field depend on the volume,

$$h_N = r_N^{-1}(1 - L^{2-d})\, L^{-(N/2)(d+2)}(\beta_N/\beta)^{1/2}$$

Theorem 1.

$$\langle \phi_k(0)\rangle \equiv \lim_{N\to\infty} \langle \phi_k(0)\rangle^{(N)} = \delta_{k1}\beta^{-1/2} \lim_{N\to\infty} L^{-(N/2)(d-2)}\beta_N^{1/2}$$

A more intuitive way to understand the above result is to go back to Eq. (1.14). For the special sequence $\{h_N\}$, $U_N(\phi, \beta, h_N)$ has a minimum at $\phi_1 = \beta_N^{1/2}$, $\phi_\perp \equiv (\phi_2,..., \phi_v) = 0$, and near the minimum

$$U_N \simeq d_N + \tfrac{1}{2}[8\lambda_N + (1 - L^{2-d})](\phi_1 - \beta_N^{1/2})^2 + \tfrac{1}{2}(1 - L^{2-d})\,\phi_\perp^2$$

Thus, the exponential in (1.14) is concentrated around $\phi = (\phi_1 = \beta_N^{1/2}, \phi_\perp = 0)$ and since $\partial R^N V/\partial\phi_k$ is zero there, only the first term survives, giving $\simeq \delta_{k1} L^{-(N/2)(d-2)}\beta^{-1/2}\beta_N^{1/2}$. This argument can be made rigorous using Lemma 5.1 of I, thus giving a direct proof of Theorem 1.

We now turn to the two-point function. Note that $\langle \phi_i(x)\,\phi_j(y)\rangle^{(N)} = 0$ for $i \neq j$. We do not know of a formula for the integrand of the numerator of Eq. (1.15), so to handle the composition $M_{R^{N-1}V}\cdots M_{R^{n-1}V}$, we derive another representation from (1.15) given by, letting $\partial_i \equiv \partial/\partial\phi_i$,

$$\begin{aligned}
&\langle \phi_i(x)\,\phi_i(y)\rangle^{(N)} \\
&= \beta^{-1} \int \Bigg\{ L^{-(N/2)(d-2)} I_i^2 + 2a_N L^{-(N/2)(d-2)} I_i \partial_i V^{(N)} \\
&\quad + a_N^2[(\partial_i V^{(N)})^2 - \partial_i^2 V^{(N)}] + \sum_{j=n-1}^{N-1} L^{-j(d-2)} \\
&\quad - (1 - L^{-d}) \sum_{j=0}^{N-n-1} a_{n+j}^2 \left(\prod_{l=N-1}^{n+j} M_{R^l V}\right) \partial_i^2 V^{(n+j)} \\
&\quad + a_{n-1}^2 L^{-d} \left(\prod_{l=N-1}^{n-1} M_{R^l V}\right) \partial_i^2 V^{(n-1)} \Bigg\}\, e^{-U_N}\, d\phi \Bigg/ \int e^{-U_N}\, d\phi
\end{aligned} \tag{1.18}$$

From (1.18) we see that to understand the two-point function we need to control products of M's applied to the second derivative of the action. Before stating the theorem on the long-range behavior of the two-point function we give an intuitive explanation: (1) $i = 1$. The first term contributes $\langle \phi_1(0)\rangle^2$ and the next three terms give a zero contribution. The dominant contribution comes from the last two terms replacing $\partial_1^2 V^{(n)}$ by

$8\lambda_n$. The term $\sum_{j=n-1}^{N-1} L^{-j(d-2)}$ gives $|x-y|^{-(d-2)}$ falloff of the inverse hierarchical Laplacian, but cancellations occur with the remaining terms to give for large d a $|x-y|^{-(d+2)}$ decay for the truncated function. (2) $i \neq 1$. The first term contributes zero and the next three terms give a zero contribution also. The dominant contribution from the last two terms comes from replacing $\partial_i^2 V^{(n)}$ by zero. The term $\sum_{j=n-1}^{N-1} L^{-j(d-2)}$ gives the dominant contribution, which is the hierarchical Laplacian decay, i.e., $|x-y|^{-(d-2)}$.

We have the following result.

Theorem 2. The thermodynamic limit of $\langle \phi_i(x)\,\phi_j(y)\rangle^{(N)}$ exists, i.e.,

$$\langle \phi_i(x)\,\phi_i(y)\rangle \equiv \lim_{N\to\infty} \langle \phi_i(x)\,\phi_i(y)\rangle^{(N)}$$

and satisfies

$$\lim_{n\to\infty} L^{(n-1)(d-2)}\langle \phi_i(x)\,\phi_i(y)\rangle = \frac{\beta^{-1}}{1-L^{-(d-2)}}, \qquad i=2, 3,..., \nu \tag{1.19}$$

$$\lim_{n\to\infty} L^{(n-1)(d-2+\delta)}(\langle \phi_1(x)\,\phi_1(y)\rangle - \langle \phi_1(0)\rangle^2) = 0 \tag{1.20}$$

where $\delta = 1/2 - 3\alpha > 0$. For large d,

$$\langle \phi_1(x)\,\phi_1(y)\rangle - \langle \phi_1(0)\rangle^2 = \beta^{-1}\frac{L^d L^{-n(d+2)}}{1-L^{-(d+2)}}\lambda^{*2}\left(\frac{1}{\lambda}+R_\infty\right)^2 + O(L^{-n(d+2+\varepsilon)})$$

where $\varepsilon > 0$, $R_\infty \equiv \lim_{n\to\infty}\ [R_n \equiv L^{2n}(\lambda_n^{-1} - \lambda^{*-1}) - (\lambda^{-1} - \lambda^{*-1})]$, and $|R_\infty| < c\beta^{\alpha-1/2}$.

Remarks. 1. To obtain the expected $|x-y|^{-(d+2)}$ decay for all $d \geqslant 3$ in Eq. (1.20), we would need more refined estimates of M operators applied to the second derivative of the effective potential. These estimates can be obtained from higher-order perturbation calculations.

2. Second-order perturbation theory shows that the dominant contribution to R_∞ is $-c(\nu-1)/\beta$, where $c > 0$.

3. The parallel truncated two-point function of the model with $V = 4\lambda(\phi_1 - \beta)^2$ can be calculated exactly and the parallel decay is given by $\beta^{-1}L^d L^{-n(d+2)}(1-L^{-(d+2)})^{-1}\ \lambda^{*2}/\lambda^2$, which is to be compared with the large-d result of Theorem 2.

We now describe the organization of this paper. In Section 2 we derive the representation of Eq. (1.18) and as an interesting byproduct we obtain a quadratic upper bound on $V^{(n)}$, $n \geqslant 1$. In Section 3 we obtain estimates on the terms of Eq. (1.18) and show the existence of the thermodynamic limit. Also, a convenient representation for the thermodynamic limit of the two-point function is obtained. Using the estimates and representation of Section 3, we obtain the decay rate of the two-point function in Section 4. We make some concluding remarks in Section 5.

2. REPRESENTATION FOR THE TWO-POINT FUNCTION

In this section we derive the representation Eq. (1.18) for the two-point function starting from Eq. (1.10). A differential inequality is obtained which gives an upper bound for $V^{(n)}(\phi)$, $n \geqslant 1$.

Set, writing $M_{R^n V} = M_{V^{(n)}} \equiv M_n$, $\partial/\partial\phi_i \equiv \partial_i$,

$$F_i^{(n-1)} \equiv -a_{n-1} L^{-(1/2)(n-1)(d-2)} I_i - a_{n-1}\, \partial_i V^{(n-1)}$$

$$G_{n,i}^{(m)} \equiv M_{n-1+m} \cdots M_n M_{n-1} (F_i^{(n-2)})^2, \qquad m \geqslant 0$$

so that

$$G_{n,i}^{(0)} = L^{-(n-1)(d-2)} M_{n-1} I_i^2 + 2a_{n-1} L^{-(1/2)(n-1)(d-2)} M_{n-1} I_i\, \partial_i V^{(n-1)} + a_{n-1}^2 M_{n-1} (\partial_i V^{(n-1)})^2$$

In order to calculate the above, we use Theorem 3. Let $n \geqslant 1$; then:

(a) $M_{n-1} I_i^2 = I_i^2 = L^{-(d-2)} I_i^2 - 2 I_i\, \partial_i V^{(n)} + L^{(d-2)}[(\partial_i V^{(n)})^2 - \partial_i^2 V^{(n)}] + 1$

(b) $M_{n-1} I_i\, \partial_i V^{(n-1)} = L^{-d} I_i\, \partial_i V^{(n)} - L^{-2}[(\partial_i V^{(n)})^2 - \partial_i^2 V^{(n)}]$

(c) $M_{n-1}[L^d (\partial_i V^{(n-1)})^2 - \partial_i^2 V^{(n-1)}] = L^{-2}[(\partial_i V^{(n)})^2 - \partial_i^2 V^{(n)}]$

Proof. (a) After making the change of variables $\xi' = L^{-(d-2)/2}\phi + \xi$ in Eq. (1.6), we find, by a direct computation, that

$$\begin{aligned} \partial_i M_{n-1} f &= L^{-(1/2)(d-2)} (M_{n-1} I_i f - L^{-(1/2)(d-2)} I_i M_{n-1} f) \\ &\quad - L^{-(1/2)(d-2)} [(M_{n-1} f)(M_{n-1} I_i) - L^{-(1/2)(d-2)} I_i M_{n-1} f] \\ &= L^{-(1/2)(d-2)} [M_{n-1} I_i f - (M_{n-1} I_i)(M_{n-1} f)] \end{aligned} \tag{2.1}$$

Setting $f = I_i$ in Eq. (2.1) and using (1.13), we get

$$M_{n-1} I_i^2 = L^{-(d-2)} I_i^2 - 2 I_i\, \partial_i V^{(n)} + L^{(d-2)}[(\partial_i V^{(n)})^2 - \partial_i^2 V^{(n)}] + 1$$

(b) Setting $f=\partial_i V^{(n-1)}$ in Eq. (2.1), we get

$$\begin{aligned} M_{n-1} I_i \partial_i V^{(n-1)} &= M_{n-1} I_i (M_{n-1} \partial_i V^{(n-1)}) + L^{(d-2)/2} \partial_i M_{n-1} \partial_i V^{(n-1)} \\ &= (L^{-(d-2)/2} I_i - L^{(d-2)/2} \partial_i V^{(n)}) L^{-(d+2)/2} \partial_i V^{(n)} \\ &\quad + L^{(d-2)/2} \partial_i L^{-(d+2)/2} \partial_i V^{(n)} \\ &= L^{-d} I_i \partial_i V^{(n)} - L^{-2}[(\partial_i V^{(n)})^2 - \partial_i^2 V^{(n)}] \end{aligned}$$

(c) Differentiating

$$\exp[-V^{(n)}(\phi)] = c_{n-1} \int \exp[-L^d V^{(n-1)}(\xi) - \tfrac{1}{2}(\xi - L^{-(d-2)/2}\phi)^2]\, d\xi$$

twice gives

$$\begin{aligned} \partial_i^2 \exp[-V^{(n)}(\phi)] &= L^{-(d-2)/2} c_{n-1} \int \left\{ \frac{\partial}{\partial \xi_i^2} \exp[-L^d V^{(n-1)}(\xi)] \right\} \\ &\quad \times \exp\left[-\frac{1}{2}(\xi - L^{-(d-2)/2}\phi)^2 \right] d\xi \end{aligned}$$

and as

$$\partial_i^2 e^{-V^{(n)}} = e^{-V^{(n)}}(\partial_i V^{(n)})^2 - e^{-V^{(n)}} \partial_i^2 V^{(n)}$$

we have

$$(\partial_i V^{(n)})^2 - \partial_i^2 V^{(n)} = L^{-(d-2)} M_{n-1}[L^{2d}(\partial_i V^{(n-1)})^2 - L^d \partial_i^2 V^{(n-1)}]$$

Using Theorem 3 in $G_{n,i}^{(0)}$ and the relation

$$a_{n+1} = L^{-(d+2)/2} a_n - L^{-(n-1)(d-2)/2} \tag{2.2}$$

we arrive at

$$\begin{aligned} G_{n,i}^{(0)} &= L^{-n(d-2)} I_i^2 + 2a_n L^{-(n/2)(d-2)} I_i \partial_i V^{(n)} + a_n^2 (\partial_i V^{(n)})^2 \\ &\quad + L^{-(n-1)(d-2)} - a_n^2 \partial_i^2 V^{(n)} + a_{n-1}^2 L^{-d} M_{n-1} \partial_i^2 V^{(n-1)} \end{aligned}$$

From this result we are led to the following.

Theorem 4. If $m \geqslant 0$,

$$\begin{aligned} G_{n,i}^{(m)} &= L^{-(n+m)(d-2)} I_i^2 + 2a_{n+m} L^{-(n+m)(d-2)/2} I_i \partial_i V^{(n+m)} \\ &\quad + a_{n+m}^2 (\partial_i V^{(n+m)})^2 \end{aligned}$$

$$+\sum_{j=n-1}^{n+m-1} L^{-j(d-2)} - a_{n+m}^2\, \partial_i^2 V^{(n+m)}$$

$$-(1-L^{-d})\sum_{j=0}^{m-1} a_{n+j}^2 \left(\prod_{l=n+n+1}^{n+j} M_l\right)$$

$$\times\, \partial_i^2 V^{(n+j)} + a_{n-1}^2 L^{-d}\left(\prod_{l=n+m-1}^{n-1} M_l\right) \partial_i^2 V^{(n-1)}$$

where the $\sum_{j=0}$ term is to be omitted for $m=0$.

Proof. The result is true for $m=0$. Assume it is true for m, $m \geqslant 0$; then, using Eq. (2.2) and Theorem 3, we have

$$\begin{aligned}
G_{n,i}^{(m+1)} &= M_{n+m} G_{n,i}^{(m)} \\
&= [L^{-(n+m-1)(d-2)} I_i^2 + 2a_{n+m+1} L^{-(n+m+1)(d-2)/2} \\
&\quad \times I_i\, \partial_i V^{(n+m+1)} + a_{n+m+1}^2 (\partial_i V^{(n+m+1)})^2] + L^{-(n+m)(d-2)} \\
&\quad - a_{n+m+1}^2\, \partial_i^2 V^{(n+m+1)} + a_{n+m}^2 L^{-d} M_{n+m}\, \partial_i^2 V^{(n+m)} \\
&\quad + \sum_{j=n-1}^{n+m-1} L^{-j(d-2)} - a_{n+m}^2 M_{n+m}\, \partial_i^2 V^{(n+m)} - (1-L^{-d}) \sum_{j=0}^{m-1} a_{n+j}^2 \\
&\quad \times \left(\prod_{l=n+m}^{n+j} M_l\right) \partial_i^2 V^{(n+j)} + a_{n-1}^2 L^{-d} \left(\prod_{l=n+m}^{n-1} M_l\right) \partial_i^2 V^{(n-1)} \\
&= [L^{-(n+m+1)(d-2)} I_i^2 + 2a_{n+m+1} L^{-(n+m+1)(d-2)/2} I_i\, \partial_i V^{(n+m+1)} \\
&\quad + a_{n+m+1}^2 (\partial_i V^{(n+m+1)})^2] \\
&\quad + \sum_{j=n-1}^{n+m} L^{-j(d-2)} - a_{n+m+1}^2\, \partial_i^2 V^{(n+m+1)} - (1-L^{-d}) \\
&\quad \times \sum_{j=0}^{n} a_{n+j}^2 \left(\prod_{l=n+m}^{n+j} M_l\right) \\
&\quad \times\, \partial_i^2 V^{(n+j)} + a_{n-1}^2 L^{-d} \left(\prod_{l=n+m}^{n-1} M_l\right) \partial_i^2 V^{(n-1)} \quad \blacksquare
\end{aligned}$$

Finally, we obtain the upper bound for $V^{(n)}$ given by the following result.

Theorem 5. For $n \geqslant 1$ and all $\phi \in R^{\nu}$,

$$V^{(n)}(\phi) \leqslant \frac{L^{-(d-2)}}{2} (|\phi| - \beta_2^{1/2})^2$$

Proof. Setting $f = I$ in Eq. (2.1) for $n \geqslant 1$ and using $(M_{n-1} I_i)^2 \leqslant M_{n-1} I_i^2$, by Schwarz's inequality, we get

$$0 \leqslant \partial_i M_{n-1} I_i = L^{-(d-2)/2} - L^{(d-2)/2}\, \partial_i^2 V^{(n)}$$

or $\partial_i^2 V^{(n)} \leqslant L^{-(d-2)}$. Since $V^{(n)}(\phi) = V^{(n)}(|\phi|)$ and $(d^p V^{(n)}/d\sigma^p)(\sigma \equiv |\phi| - \beta_n^{1/2}) = 0$, for $p = 0, 1$ the result follows.

3. TWO-POINT FUNCTION AND THERMODYNAMIC LIMIT

In this section, we establish a convenient representation for functions of the form $(\prod_{l-n+m}^{n} M_l)\, \partial_i^2 V^{(n)}$ appearing in Theorem 4, which will ensure the control of the integral (1.18); the infinite-volume two-point function will then be written as a superposition of multiscale contributions analogous to the expansion of the free energy obtained in I.

The integrals in (1.18) will be calculated splitting the region of integration, as in I, into "small fields" (perturbative region)

$$\{\phi\colon |\phi_1 - \beta_N^{1/2}| < \tfrac{1}{4}\beta_N^{\alpha},\ |\phi_\perp| = (\phi_2^2 + \cdots + \phi_\nu^2)^{1/2} < \tfrac{1}{4}\beta_N^{\alpha}\}$$

and the complementary "large-fields" region; $0 < \alpha < 1/6(d-2)$ is a fixed number. The latter contribution can be handled with a crude global upper bound on $(\prod_{l-n+m}^{n} M_l)\, \partial_i^2 V^{(n)}$, given in Theorem 6 below. The main contribution comes from the perturbative region, where a detailed representation, given in Theorem 7, is needed.

Let $n \geqslant 1$ and $m \geqslant -1$. When $m = -1$, $(\prod_{l=n-1}^{n} M_l)\, \partial_i^2 V^{(n)} \equiv \partial_i^2 V^{(n)}$. The constants appearing in the theorems below depend only on L and α. We assume the initial β to be large (depending on L and α) and the initial $\lambda > \tfrac{1}{2}\lambda^*$.

Theorem 6. The function

$$e^{-2V^{(n+m+1)}(\phi)} \left(\prod_{l=n+m}^{n} M_l\right) \partial_i \partial_j V^{(n)}(\phi)$$

is entire in ϕ and bounded by $d_0 \beta_{n+m}^{\nu} [\exp L^{-(d-2)} (\operatorname{Im} \phi)^2]$, for a suitable constant d_0. For real vectors,

$$\left| e^{-V^{(n+m+1)}(\phi)} \left(\prod_{l=n+m}^{n} M_l\right) \partial_i^2 V^{(n)}(\phi) \right| \leqslant d_0 \beta_{n+m}$$

To obtain the small-fields representation for $(\prod_{l-n+m}^{n} M_l)\ \partial_i \partial_j V^{(n)}$, we first note that since it is a second-rank tensor under rotations, it has the general form

$$\left(\prod_{l=n+m}^{n} M_l\right) \partial_i \partial_j V^{(n)}(\phi) = f_m^{(n)}(|\phi|)\, \delta_{ij} + \frac{\phi_i \phi_j}{|\phi|^2}\, g_m^{(n)}(|\phi|)(|\phi|) \qquad (3.1)$$

Theorem 7. $f_m^{(n)}$ and $h_m^{(n)} \equiv f_m^{(n)} + g_m^{(n)}$, viewed as functions of $\sigma = |\phi| - \beta_{n+m+1}^{1/2}$, are analytic on $|\sigma| < \frac{1}{2}\beta_{n+m+1}^{\alpha}$ and have the representation $f_m^{(n)}(|\phi|) = \gamma_m^{(n)} + \tilde{f}_m^{(n)}(\sigma)$ and $h_m^{(n)}(|\phi|) = 8\lambda_m^{(n)} + \tilde{h}_m^{(n)}(\sigma)$, where $\tilde{f}_m^{(n)}(\sigma=0) = \tilde{h}_m^{(n)}(\sigma=0) = 0$ and $|\tilde{f}_m^{(n)}(\sigma)|$, $|\tilde{h}_m^{(n)}(\sigma)| \leqslant k\beta_{n+m+1}^{(3\alpha-1/2)}$. Also, $\gamma_{-1}^{(n)} = 0$ and $\lambda_{-1}^{(n)} = \lambda_n$, and we have $|\gamma_{m+1}^{(n)} - \gamma_m^{(n)}|$, $8\,|\lambda_{m+1}^{(n)} - \lambda_m^{(n)}| \leqslant k\beta_{n+m+1}^{(3\alpha-1/2)}$.

Before we prove Theorems 6 and 7, we establish the final representation for the thermodynamic limit of the two-point function which is used to analyze the decay in Section 4.

Theorem 8. The infinite-volume truncated two-point function is given by

$$\begin{aligned}
&\beta[\langle\phi_i(x)\,\phi_i(y)\rangle - \langle\phi_i(x)\rangle\langle\phi_i(y)\rangle] \\
&\quad= \frac{L^{-(n-1)(d-2)}}{1-L^{-(d-2)}} - (1-L^{-d}) \\
&\qquad\times \sum_{j=0}^{\infty} a_{n+j}^2[8\lambda_{n+j}\delta_{i1} + O(\beta_{n+j}^{3\alpha-1/2})] \\
&\qquad+ L^{-d}a_{n-1}^2[8\lambda_{n-1}\delta_{i1} + O(\beta_{n-1}^{3\alpha-1/2})]
\end{aligned}$$

where $n = n(x, y)$ defines the hierarchy containing $x, y \in Z^d$.

Proof. Rewrite Eq. (1.18) as

$$\langle\phi_i(x)\,\phi_i(y)\rangle^{(N)} = \frac{N_n^{(N)}}{D^{(N)}}$$

where

$$\begin{aligned}
N_n^{(N)} = \int \Bigg\{ & L^{-N(d-2)}\phi_i^2 + 2a_N L^{-N(d-2)/2}\phi_i\,\partial_i V^{(N)} \\
&+ a_N^2[(\partial_i V^{(N)})^2 - \partial_i^2 V^{(N)}] \\
&+ \sum_{j=n-1}^{N-1} L^{-j(d-2)} - (1-L^{-d}) \\
&\times \sum_{j=0}^{N-n-1} a_{n+j}^2 \left(\prod_{l=N-1}^{n+j} M_l\right) \partial_i^2 V^{(n+j)} \\
&+ L^{-d}a_{n-1}^2 \left(\prod_{l=N-1}^{n-1} M_l\right) \partial_i^2 V^{(n-1)} \Bigg\} e^{-\tilde{U}_N}\, d\phi
\end{aligned}$$

$$D^{(N)} = \int e^{-\tilde{U}_N}\, d\phi$$

where

$$\tilde{U}_N(\phi) = V^{(N)}(\phi) + \tfrac{1}{2}(1 - L^{2-d})[(\phi_1 - \beta_N^{1/2})^2 + \phi_\perp^2] = U_N(\phi, h_N)$$

The particularly simple form of $\tilde{U}_N$ is a consequence of the judicious choice of the sequence h_N of magnetic fields in I. To calculate $D^{(N)}$, we split the integral according to the decomposition $1 = \chi_o + \chi_c$, where

$$\chi_o(\phi) = \begin{cases} 1 & \text{if } |\phi_1 - \beta_N^{1/2}| < \tfrac{1}{4}\beta_N^\alpha \text{ and } |\phi_\perp| < \tfrac{1}{4}\beta_N^\alpha \\ 0 & \text{otherwise} \end{cases}$$

Since $V^{(N)} \geqslant 0$,[1] we have

$$|D_c^{(N)}| = \left| \int \chi_c \exp(-\tilde{U}_N)\, d\phi \right| \leqslant \text{const} \times \exp\left[-\frac{1}{64}(1 - L^{2-d})\, \beta_N^\alpha \right]$$

Using the "small-field" representation

$$V^{(N)}(\phi) = 4\lambda_N(|\phi| - \beta_N^{1/2})^2 + w_N(|\phi| - \beta_N^{1/2})$$

on $||\phi| - \beta_N^{1/2}| < \beta_N^\alpha$, with w_N analytic and vanishing together with the first two derivatives at zero and bounded by $k\beta_N^{(3\alpha - 1/2)}$ there, we can write $\tilde{U}_N(\phi) = v_N + O(\beta_N^{(3\alpha - 1/2)})$, where

$$v_N(\phi) \equiv \tfrac{1}{2}[8\lambda_N + (1 - L^{2-d})](\phi_1 - \beta_N^{1/2})^2 + \tfrac{1}{2}(1 - L^{2-d})\, \phi_\perp^2$$

valid if $\chi_o(\phi) = 1$, with the $O(\cdot)$ term uniform in this region. Thus,

$$D_o^{(N)} = \int \chi_o e^{-\tilde{U}_N}\, d\phi = \int \exp[-V_N(\phi)]\, d\phi + O(\beta_N^{3\alpha - 1/2})$$

and we conclude that

$$\lim_{n \to \infty} D^{(N)} \equiv D^{(\infty)} = \int \exp\{-\tfrac{1}{2}[8\lambda + (1 - L^{2-d})]\, \phi_1^2 - \tfrac{1}{2}(1 - L^{2-d})\, \phi_1^2\}\, d\phi$$

We analyze each term of $N_n^{(N)}$ in a similar way. Thus, we find

$$\int \phi_i^2 \chi_c \exp(-\tilde{U}_N)\, d\phi \leqslant \text{const} \times (\beta\delta_{i1} + \text{const}) \times \exp\left[-\frac{1}{64}(1 - L^{2-d})\, \beta_N^{2\alpha} \right]$$

and

$$\int \phi_i^2 \chi_o e^{-\tilde{U}_N}\, d\phi = [1 + O(\beta_N^{3\alpha - 1/2})] \int \phi_i^2 \chi_o e^{-V_N}\, d\phi$$

Hence

$$\int \phi_i^2 \chi_o \exp(-\tilde{U}_N)\, d\phi$$

$$= [1 + O(\beta_N^{3\alpha - 1/2})] \left\{ \int \phi_i^2 \exp(-V_N)\, d\phi + O\left(\beta_N \exp\left[-\frac{1}{64}(1 - L^{2-d})\,\beta_N^{2\alpha}\right]\right)\right\}$$

$$= \left[\int \exp(-v_N)\, d\phi\right] [1 + O(\beta^{3\alpha - 1/2})] \times [\beta_N \delta_{i1} + O(1)]$$

It follows that

$$\lim_{N \to \infty} \int L^{-N(d-2)} \phi_i^2 e^{-\tilde{U}_N}\, d\phi = \delta_{i1} D^{(\infty)} \lim_{N \to \infty} L^{-N(d-2)} \beta_N$$
$$= D^{(\infty)} \langle \phi_i(0) \rangle^2$$
$$= D^{(\infty)} \langle \phi_i(x) \rangle \langle \phi_i(y) \rangle$$

Next consider

$$\int \phi_i\, \partial_i V^{(N)}\, e^{-\tilde{U}_N}\, d\phi = -\int (\partial_i e^{-V^{(N)}})\, \phi_i e^{-u_N}\, d\phi$$

with

$$u_N(\phi) = \tfrac{1}{2}(1 - L^{2-d})[(\phi_1 - \beta_N^{1/2})^2 + \phi_\perp^2]$$

We have

$$\left| \int \phi_i\, \partial_i V^{(N)}\, e^{-\tilde{U}_N}\, \partial\phi \right|$$
$$= \left| \int e^{-V^{(N)}} e^{-u_N}\, d\phi - \int e^{-V^{(N)}} \phi_i (\partial_i u_n)\, e^{-u_N}\, \partial\phi \right|$$
$$\leqslant \text{const} + \delta_{i1} \beta^{1/2} \cdot \text{const}$$

Since $a_N \sim L^{-N(d-2)/2}$, we conclude that

$$\lim_{N \to \infty} a_N L^{-N(d-2)/2} \int \phi_i\, \partial_i V^{(N)}\, e^{-\tilde{U}_N}\, d\phi = 0$$

Now, consider

$$\left|\int [(\partial_i V^{(N)})^2 - \partial_i^2 V^{(N)}]\, e^{-V^{(N)} - u_N}\, d\phi\right|$$
$$= \left|\int (\partial_i^2 e^{-V^{(N)}})\, e^{-u_N}\, d\phi\right|$$
$$= \left|\int e^{-V^{(N)}}\, \partial_i^2 e^{-u_N}\, d\phi\right|$$
$$= \left|\int [(\partial_i u_N)^2 - \partial_i^2 u_N]\, e^{-V^{(N)} - u_N}\, d\phi\right|$$
$$\leqslant \text{const}$$

which implies

$$\lim_{n \to \infty} a_N^2 \int [(\partial_i V^{(N)})^2 - \partial_i^2 V^{(N)}]\, e^{-\tilde{U}_N}\, d\phi = 0$$

Next, using Theorem 6, we have

$$\left|\int \left(\prod_{l=N-1}^{n+j} M_l\right) \partial_i^2 V^{(n+j)} [\exp(-\tilde{U}_N)]\, \chi_c\, d\phi\right|$$
$$\leqslant \text{const} \times \beta_{N-1-n-j}^{v} \exp\left[-\frac{1}{64}(1 - L^{2-d})\, \beta_N^{2\alpha}\right]$$

Also, since $||\phi| - \beta_N^{1/2}| < \frac{1}{2}\beta_N^{\alpha}$ if $\chi_o(\phi) = 1$, we have, from Theorem 7,

$$\int \left(\prod_{l=N-1}^{n+j} M_l\right) \partial_i^2 V^{(n+j)}\, e^{-\tilde{U}_N} \chi_o\, d\phi$$
$$= \int \left[\left(1 - \frac{\phi_i^2}{|\phi|^2}\right) f_{N-1-n-j}^{(n+j)}(|\phi|) + \frac{\phi_i^2}{|\phi|^2} h_{N-1-n-j}^{(n+j)}(|\phi|)\right] e^{-v_N} \chi_o\, d\phi + O(\beta_N^{3\alpha - 1/2})$$
$$= \gamma_{N-1-n-j}^{(n+j)} \int e^{-v_N}\, d\phi + (8\lambda_{N-1-n-j}^{(n+j)} - \gamma_{N-1-n-j}^{(n+j)}) \times \int \frac{\phi_i^2}{|\phi|^2} e^{-v_N} \chi_o\, d\phi + O(\beta_N^{3\alpha - 1/2})$$

Since $\phi_i^2/|\phi|^2 = \delta_{i1} + O(\beta_N^{2\alpha - 1})$ if $\chi_o(\phi) = 1$, the above expression is equal to

$$[8\lambda_{N-1-n-j}^{(n+j)} \delta_{i1} + \gamma_{N-1-n-j}^{(n+j)} (1 - \delta_{ij})] \int e^{-v_N}\, d\phi + O(\beta_N^{3\alpha - 1/2})$$

Now, Theorem 7 implies that $\lim_{m\to\infty}\gamma_m^{(n)}=\gamma_\infty^{(n)}$ and $\lim_{m\to\infty}\lambda_m^{(n)}\equiv\lambda_\infty^{(n)}$ exists and $\gamma_\infty^{(n)}=O(\beta_n^{3\alpha-1/2})$ and $\lambda_\infty^{(n)}=\lambda_n+O(\beta_n^{3\alpha-1/2})$.

Thus, we conclude that

$$\lim_{N\to\infty}\int\left(\prod_{l=N-1}^{n+j} M_l\right)\partial_i V^{2(n+j)}\,e^{-\tilde{U}_N}\,d\phi$$
$$=[8\lambda_\infty^{(n+j)}\delta_{i1}+\lambda_\infty^{(n+j)}(1-\delta_{i1})]\,D^{(\infty)}$$

and in the same way,

$$\lim_{N\to\infty}\left(\prod_{l=N-1}^{n-1} M_l\right)\partial_i^2 V^{(n-1)}\,e^{-\tilde{U}_N}\,d\phi$$
$$=[8\lambda_\infty^{(n-1)}\delta_{i1}+\gamma_\infty^{(n-1)}(1-\delta_{i1})]\,D^{(\infty)}$$

Since

$$\left|\int\left(\prod_{l=N-1}^{n+j} M_l\right)\partial_i^2 V^{(n+j)}\,e^{-\tilde{U}_N}\,d\phi\right|\leqslant O(1)\qquad\text{as}\quad N\to\infty$$

independently of j, the results established above together with the dominated convergence theorem complete the proof of Theorem 8.

Proof of Theorem 6. We first verify the assertion for $m=-1$.

From (2.1) and (1.13) we have the following identity valid if $n\geqslant 1$:

$$\partial_i\partial_j V^{(n)}=L^{-(d-2)}[\delta_{ij}-M_{n-1}I_iI_j+(M_{n-1}I_i)(M_{n-1}I_j)]$$

In I, we have shown that

$$\exp[-V^{(n)}(\phi)]=c_{n-1}\int\exp[-L^dV^{(n-1)}(\xi)-\tfrac{1}{2}(\xi-L^{-(d-2)/2}\phi)^2]\,d\xi$$

with

$$c_{n-1}=(1+8\lambda_{n-1}L^d)^{1/2}\,[1+O(\beta_n^{3\alpha-1/2})]$$

[recall that $d\xi$ includes a $(2\pi)^{-\nu/2}$ factor], so that clearly $e^{-2V^{(n)}}\partial_i\partial_j V^{(n)}$ is an entire function. To estimate it, we consider the case $n=1$ separately from $n>1$. Using the fact that

$$V^{(0)}(\xi)=\frac{\lambda}{\beta}(\xi^2-\beta)^2\geqslant\lambda(|\xi|-\beta^{1/2})^2$$

we have (d_1, d_2 are constants depending on L and v)

$$|e^{-V^{(1)}} M_0 I_i|$$
$$\leqslant C_0 d_1 \exp[\tfrac{1}{2} L^{-(d-2)} (\operatorname{Im} \phi)^2 - \tfrac{1}{2} L^{-(d-2)} (\operatorname{Re} \phi)^2 - \lambda L^d \beta]$$
$$\times \int_0^\infty r^v \exp[-\tfrac{1}{2}(1 + 2\lambda L^d) r^2 + (2\lambda L^d \beta^{1/2} + L^{-(d-2)/2} |\operatorname{Re} \phi|) r] \, dr$$

Proceeding as in the proof of the global upper bound in I, we get

$$|e^{-V^{(1)}} M_0 I_i|$$
$$\leqslant \frac{C_0 d_2}{(1 + 2\lambda L^d)^{1/2}} (\beta^{1/2} + L^{-(d-2)/2} |\operatorname{Re} \phi|)^v$$
$$\times \exp\left[\frac{1}{2} L^{-(d-2)} (\operatorname{Im} \phi)^2 - \frac{\lambda L^d}{1 + 2\lambda L^d} (|\operatorname{Re} \phi| - L^{(d-2)/2} \beta^{1/2})^2\right]$$

Since $\lambda \geqslant \frac{1}{2}\lambda^* = (L^2 - 1)/16 L^d$, we get

$$|[\exp(-V^{(1)})] \, M_0 I_i| \leqslant d_3 \beta^{v/2} \exp[\tfrac{1}{2} L^{-(d-2)} (\operatorname{Im} \phi)^2]$$

Similarly,

$$|[\exp(-V^{(1)})] \, M_0 I_i I_j| \leqslant d_4 \beta^{(v+1)/2} \exp[\tfrac{1}{2} L^{-(d-2)} (\operatorname{Im} \phi)^2]$$

Using also the fact that

$$|\exp(-V^{(1)})| \leqslant \exp[-\lambda(|\operatorname{Re} \phi| - \beta_1)^2 + \tfrac{1}{2} L^{-(d-2)} (\operatorname{Im} \phi)^2]$$

we finally get

$$|[\exp(-2V^{(1)})] \, \partial_i \partial_j V^{(1)}| \leqslant d_0 \beta^v \exp[L^{-(d-2)} (\operatorname{Im} \phi)^2]$$

If $n > 1$, we use $V^{(n-1)}(\xi) \geqslant \lambda(|\xi| - \beta_{n-1}^{1/2})^2$ and $\lambda_{n-1} \leqslant 3\lambda$ established in I. The calculations are essentially the same as in the $n = 1$ case and we again get

$$|[\exp(-2V^{(n)})] \, \partial_i \partial_j V^{(n)}| \leqslant d_0 \beta_{n-1}^v \exp[L^{-(d-2)} (\operatorname{Im} \phi)^2]$$

Now, assume the assertion holds if $m \geqslant -1$; then which shows that

$$[\exp(-2V^{(n+m+2)})] \left(\prod_{l=n+m+1}^{n} M_l \right) \partial_i \partial_j V^{(n)}$$
$$= [\exp(-V^{(n+m+2)})] \, C_{n+m+1}$$
$$\times \int \left(\prod_{l=n+m}^{n} M_l \right) \partial_i \partial_j V_j^{(n)}(\xi)$$
$$\times \exp[-L^d V^{(n+m+1)}(\xi) - \tfrac{1}{2}(\xi - L^{-(d-2)/2} \phi)^2] \, d\xi$$

which shows that the left-hand side above is entire. Also, from the inductive bound,

$$\left| e^{-2V^{(n+m+2)}} \left(\prod_{l=n+m+1}^{n} M_l \right) \partial_i \partial_j V^{(n)} \right| \leqslant d_0 \beta^{\nu}_{n+m} c_{n+m+1} e^{L^{-(d-2)}(\mathrm{Im}\,\phi)^2}$$

and $c_{n+m+1} \leqslant 2(1+24\lambda L^d)^{1/2} \leqslant 6L$. Using the fact that $\beta^{\nu}_{n+m} \leqslant 2L^{-\nu(d-2)}\beta^{\nu}_{n+m+1}$, we see that the right-hand side is bounded by $d_0 \beta^{\nu}_{n+m+1} e^{L^{-(d-2)}(\mathrm{Im}\,\phi)^2}$ and completes the proof of the theorem in the complex case. If ϕ is real, we have

$$|\partial_i^2 V^{(n)}| = L^{-(d-2)}\, |1 + (M_{n-1} I_i)^2 - M_{n-1} I_i^2| \leqslant L^{-(d-2)}(1 + 2M_{n-1} I_i^2)$$

from which we get $|e^{-V^{(n)}} \partial_i^2 V^{(n)}| \leqslant d\beta_{n-1}^{(\nu+1)/2} < d_0 \beta^{\nu}_{n-1}$. The induction in m now proceeds as before. ∎

Proof of Theorem 7. We first verify the assertion when $m \equiv -1$. From I, we know that

$$V^{(n)}(\phi) = 4\lambda_n \sigma^2 + w_n(\sigma) \qquad \text{on} \quad |\sigma| \equiv ||\phi| - \beta_n^{1/2}| < \beta_n^{\alpha}$$

with w_n analytic and vanishing together with the first two derivatives at zero, and bounded by $k\beta_n^{3\alpha - 1/2}$. We use the notation $|\phi| = (\phi_1^2 + \cdots + \phi_\nu^2)^{1/2}$ even for complex ϕ, $|\phi|$ is analytic in a suitable region. We have

$$\partial_i \partial_j V^{(n)}(\phi) = \frac{1}{|\phi|} \frac{dV^{(n)}}{d\sigma} \sigma_{ij} + \frac{\phi_i \phi_j}{|\phi|^2} \left(\frac{d^2 V^{(n)}}{d\sigma^2} - \frac{1}{|\phi|} \frac{dV^{(n)}}{d\sigma} \right)$$

showing that

$$f^{(n)}_{-1}(|\phi|) = \frac{1}{\sigma + \beta_n^{1/2}} \left(8\lambda_n \sigma + \frac{dw_n}{d\sigma} \right), \qquad h^{(n)}_{-1}(|\phi|) = 8\lambda_n + \frac{d^2 w_n}{d\sigma^2}$$

Thus, $\gamma^{(n)}_{-1} = 0$ and $\lambda^{(n)}_{-1} = \lambda_n$. If we restrict $|\sigma| < \frac{1}{2}\beta_n^{\alpha}$, we have by a Cauchy estimate

$$|\tilde{f}^{(n)}_{-1}(|\phi|)| \leqslant \frac{2}{\beta_n^{1/2}} (12\lambda \beta_n^{\alpha} + 4k\beta_n^{(2\alpha - 1/2)}) \leqslant k\beta_n^{(3\alpha - 1/2)}$$

(we take without loss of generality $k \geqslant 1$). Similarly,

$$|\tilde{h}^{(n)}_{-1}(|\phi|) \leqslant 16k\beta_n^{(\alpha - 1/2)} < k\beta_n^{(3\alpha - 1/2)}$$

Now, assume the assertion true for $m \geqslant -1$ and compute. Thus,

$$f^{(n)}_{m+1}(|\phi|) = \left(\prod_{l=n+m+1}^{n} M_l \right) \partial_2^2 V^{(n)}(|\phi|\, \hat{e}_1)$$

$$h^{(n)}_{m+1}(|\phi|) = \left(\prod_{l=n+m+1}^{n} M_l \right) \partial_1^2 V^{(n)}(|\phi|\, \hat{e}_1)$$

We have

$$f^{(n)}_{m+1}(|\phi|)$$

$$= \left\{ \int \left(\prod_{l=n+m+1}^{n} M_l \right) \partial_2^2 V^{(n)}(\xi) \right.$$

$$\left. \times \exp[-L^d V^{(n+m+1)}(\xi) - \tfrac{1}{2}(\xi - L^{-(d-2)/2} |\phi| \hat{e}_1)^2] \, d\xi \right\}$$

$$\times \left\{ \int \exp[-L^d V^{(n+m+1)}(\xi) - \tfrac{1}{2}(\xi - L^{-(d-2)/2} |\phi| \hat{e}_1)^2] \, d\xi \right\}^{-1} \tag{3.2}$$

The denominator is proportional to $\exp[-V^{(n+m+2)}(\phi)]$ and is nonzero on $||\phi| - \beta^{1/2}_{n+m+2}| < \beta^{\alpha}_{n+m+2}$. Letting $\sigma' = |\phi| - L^{(d-2)/2}\beta^{1/2}_{n+m+1}$ and using the fact that $\beta^{1/2}_{n+m+2} = L^{(d-2)/2}\beta^{1/2}_{n+m+1} + O(\beta^{2\alpha-1/2}_{n+m+1})$ established in I, it follows that $f^{(n)}_m$ is analytic on $|\sigma'| < \frac{1}{2}(2L^{(d-2)}\beta_{n+m+1})^{\alpha}$. Using Theorem 6, we make a complex shift in (3.2), $\xi \to \xi + L^{-(d-2)/2} |\phi| \hat{e}_1$, and write $\xi = u\hat{e}_1 + t$, with $t \cdot \hat{e}_1 = 0$. Next, define

$$\tilde{V}^{(n+m+1)}[(u + L^{-(d-2)/2}\sigma' + \beta^{1/2}_{n+m+1}) \hat{e}_1 + t]$$
$$= V^{(n+m+1)}[(u + L^{-(d-2)/2}\sigma' + \beta^{1/2}_{n+m+1}) \hat{e}_1 + t]$$
$$- 4\lambda_{n+m+1}(u + L^{-(d-2)/2}\sigma')^2$$

and expand the resulting quadratic form in u around the point where its first derivative vanishes. Finally, making a second complex shift

$$u \to u - \frac{8\lambda_{n+m+1} L^{(d+2)/2}}{1 + 8\lambda_{n+m+1} L^d} \sigma'$$

we arrive at

$$f^{(n)}_{m+1}(|\phi|)$$

$$= \left\{ \int \left(\prod_{l=n+m+1}^{n} M_l \right) \partial_2^2 V^{(n)}(r) \right.$$

$$\left. \times \exp[-L^d V^{(n+m+1)}(r) - \tfrac{1}{2}(1 + 8\lambda_{n+m+1} L^d) \, u^2 - \tfrac{1}{2}t^2] \, du \, dt \right\}$$

$$\times \left\{ \int \exp[-L^d V^{(n+m+1)}(r) - \tfrac{1}{2}(1 + 8\lambda_{n+m+1} L^d) \, u^2 - \tfrac{1}{2}t^2] \, du \, dt \right\}^{-1}$$

$$= \frac{N}{D} \tag{3.3}$$

where

$$r = \left(\frac{L^{-(d-2)/2}\sigma'}{1 + 8\lambda_{n+m+1} L^d} + u + \beta_{n+m+1}^{1/2} \right) \hat{e}_1 + t$$

and the t integration in (3.3) is over the $(v-1)$-dimensional space orthogonal to the $\hat{e}_1$ direction.

We analyze (3.3) in a way similar to the construction in I. The region of small and large fluctuation fields is specified by the characteristic function

$$\chi_o(u, t) = \begin{cases} & \text{if} \quad |u|, |t| < L^{-(d-1/3)/2} \beta_{n+m+1}^{\alpha} \\ & \text{otherwise} \end{cases}$$

and we write $N = N_o + N_c$ and $D = D_o + D_c$, corresponding to the decomposition $1 = \chi_o + \chi_x$. From the global upper bound of Theorem 6,

$$\begin{aligned} |N_c| = &\left| \int \left(\prod_{l=n+m+1}^{n} M_l \right) \partial_2^2 V^{(n)}(r) \right. \\ &\times \exp\left[-L^d V_{n+m+1}(r) + 4\lambda_{n+m+1} L^d \left(u + \frac{L^{-(d-2)/2}\sigma'}{1 + 8\lambda_{n+m+1} L^d} \right)^2 \right. \\ &\left.\left. - \frac{1}{2} (1 + 8\lambda_{n+m+1} L^d)\, u^2 - \frac{1}{2} t^2 \right] \chi_c(u, t)\, du\, dt \right| \\ \leqslant\; & d_0 \beta_{n+m}^{v} \int \exp\left[-L^d (1 - 2L^d)\, \lambda (|\text{Re}\, r| - \beta_{n+m+1}^{1/2})^2 \right. \\ &+ \frac{1}{2} L^2 (\text{Im}\, r)^2 + 4\lambda_{n+m+1} L^d\, \text{Re} \left(u + \frac{L^{-(d-2)/2}\sigma'}{1 + 8\lambda_{n+m+1} L^d} \right)^2 \\ &\left. - \frac{1}{2} (1 + 8\lambda_{n+m+1} L^d)\, u^2 - \frac{1}{2} t^2 \right] \chi_c\, du\, dt \end{aligned} \tag{3.4}$$

Now

$$\begin{aligned} &\frac{1}{2} L^2 (\text{Im}\, r)^2 + 4\lambda_{n+m+1} L^d\, \text{Re} \left(u + \frac{L^{-(d-2)/2}\sigma'}{1 + 8\lambda_{n+m+1} L^d} \right)^2 - 4\lambda_{n+m+1} L^d u^2 \\ &\quad \leqslant \frac{1}{2} L^{4-d} |\sigma'|^2 + \frac{8\lambda_{n+m+1} L^{(d+2)/2}}{1 + 8\lambda_{n+m+1} L^d}\, u\, \text{Re}\, \tau \\ &\quad \leqslant L^{4-d} |\sigma'|^2 + \frac{1}{9} (u^2 + t^2) \\ &\quad \leqslant \delta(L)\, \beta_{n+m+1}^{2\alpha} + \frac{1}{9} (u^2 + t^2) \end{aligned}$$

where $\delta(L)=\frac{1}{4}L^{4-d}(2-L^{d-2})^{2\alpha}$. Hence, if $|u|$ or $|t|\geqslant(12\delta)^{1/2}\,\beta^{\alpha}_{n+m+1}$, the exponent in (3.4) is bounded above by $-\frac{1}{4}(u^2+t^2)$.

Now, suppose $|u|$ and $|t|\leqslant(12\delta)^{1/2}\,\beta^{\alpha}_{n+m+1}$, but still $|u|$ or $|t|\geqslant L^{-(d-1/3)/2}\beta^{\alpha}_{n+m+1}$. Then

$$|\mathrm{Re}\, r|-\beta^{1/2}_{n+m+1}=\left(u+\frac{L^{-(d-2)/2}}{1+8\lambda_{n+m+1}L^d}\,\mathrm{Re}\,\sigma'\right)^1_2+\beta^{-1/2}_{n+m+1}t^2+O(\beta^{3\alpha-1}_{n+m+1})$$

which implies, after some algebra, that

$$\begin{aligned}
&-L^d(1-2L^{-d})\,\lambda(|\mathrm{Re}\,r|-\beta^{1/2}_{n+m+1})^2+\frac{1}{2}(\mathrm{Im}\,r)^2+4\lambda_{n+m+1}L^d\\
&\quad\times\mathrm{Re}\left(u+\frac{L^{-(d-2)/2}\sigma'}{(1+8\lambda_{n+m+1}L^d)^2}\right)^2-4\lambda_{n+m+1}L^du^2\\
&\quad-\frac{[4\lambda_{n+m+1}-(1-2L^{-d})\lambda]\,L^2}{(1+8\lambda_{n+m+1}L^d)^2}(\mathrm{Re}\,\sigma')^2\\
&\quad+\frac{(L^{2-d}-8\lambda_{n+m+1})\,L^2}{2(1+8\lambda_{n+m+1}L^d)^2}(\mathrm{Im}\,\sigma')^2\\
&\quad+\frac{2[4\lambda_{n+m+1}-(1-2L^{-d})\lambda]\,L^{(d+2)/2}}{(1+8\lambda_{n+m+1}L^d)}u(\mathrm{Re}\,\sigma')\\
&\quad-L^d(1-2L^{-d})\lambda u^2+O(\beta^{3\alpha-1/2}_{n+m+1})\\
&\leqslant\frac{4\lambda_{n+m+1}}{(1-2L^d)\lambda}\,\frac{[4\lambda_{n+m+1}-(1-2L^{-d})\lambda]\,L^2}{(1+8\lambda_{n+m+1}L^d)^2}(\mathrm{Re}\,\sigma')^2\\
&\quad+\frac{(L^{2-d}-8\lambda_{n+m+1})\,L^2}{2(1+8\lambda_{n+m+1}L^d)^2}(\mathrm{Im}\,\sigma')^2+O(\beta^{3\alpha-1/2}_{n+m+1})
\end{aligned}\tag{3.5}$$

Using $\frac{1}{2}\lambda^*\leqslant\lambda_n\leqslant 3\lambda^*$ if $n\geqslant 1$, the coefficient of $(\mathrm{Re}\,\sigma')^2$ in (3.5) is bounded by $288\lambda L^2/(1+4\lambda L^d)^2$, which is smaller than $144/L^d$ since $\lambda^*=(L^2-1)/8L^d$. Likewise, the coefficient of $(\mathrm{Im}\,\sigma')^2$ is bounded by $2L^{-d}$. Thus, the exponent in (3.4) is bounded by

$$\begin{aligned}
&\frac{144}{L^d}\,|\sigma'|^2-\frac{1}{2}(u^2+t^2)+O(\beta^{3\alpha-1/2}_{n+m+1})\\
&\leqslant 40L^{-d}(2L^{d-2}\beta_{n+m+1})^{2\alpha}-\frac{1}{2}(u^2+t^2)\\
&\leqslant\left[\frac{40(2L^{(d-2)})^{2\alpha}}{L^{1/3}}-\frac{1}{2}\right](u^2+t^2)
\end{aligned}$$

because $|u|$ or $|t| \geqslant L^{-(d-1/3)/2}\beta^{\alpha}_{n+m+1}$. We choose α small and L large so that $(2L^{(d-2)})^{2\alpha}/L^{1/3} \leqslant 1/160$. This implies that the exponent in (3.4) is bounded by $(-1/4)(u^2+t^2)$ for all (u, t) such that $\chi_c \equiv 1$. Therefore, we conclude that

$$|N_c| \leqslant \text{const} \cdot \beta^{\nu}_{n+m} \exp[-\tfrac{1}{8}L^{-(d-1/3)}\beta^{2\alpha}_{n+m+1}]$$

In an analogous way, we find $|D_c| \leqslant \text{const} \cdot \exp[-\frac{1}{8}L^{-(d-1/3)}\beta^{2\alpha}_{n+m+1}]$.

We now turn to the calculation of

$$N_o = \int \left(\prod_{l=n+m}^{n} M_l \right) \partial_2^2 V^{(n)}(r) \exp\{-L^d \tilde{V}^{(n+m+1)}(r) - \tfrac{1}{2}(1 + 8\lambda_{n+m+1}L^d)\, u^2 - \tfrac{1}{2}t^2\}\, \chi_o \, du\, dt$$

Because of the χ_o function, we have

$$\eta = |r| - \beta^{1/2}_{n+m+1} = \left(u + \frac{L^{-(d-2)/2}\sigma'}{1+8\lambda_{n+m+1}L^d}\right) + \frac{1}{2}\beta^{-1/2}_{n+m+1}t^2 + O(\beta^{3\alpha-1}_{n+m+1})$$

Using again the fact that $\frac{1}{2}\lambda^* \leqslant \lambda_n \leqslant 3\lambda^*$ if $n \geqslant 1$, we see that

$$\left| u + \frac{L^{-(d-2)/2}\sigma'}{1+8\lambda_{n+m+1}L^d} \right| \leqslant 2L^{(d-1/3)/2}\beta^{\alpha}_{n+m+1}, \qquad |\eta| \leqslant 3L^{-(d-1/3)/2}\beta^{\alpha}_{n+m+1}$$

Hence, we can use the small-fields representation for $V^{(n+m+1)}(r)$ to get

$$\tilde{V}^{(n+m+1)}(r) = 4\lambda_{n+m+1}\beta^{-1/2}_{n+m+1}\left(u + \frac{L^{-(d-2)/2}}{1+8\lambda_{n+m+1}L^d}\sigma'\right)t^2 + w_{n+m+1}(\eta) + O(\beta^{4\alpha-1}_{n+m+1})$$

The first term in the right-hand side is bounded by $3L^{-(5/2)(d-1)}\beta^{3\alpha-1/2}_{n+m+1}$. To estimate $w_{n+m+1}(\eta)$, we note since $\sigma^{-3}w_{n+m+1}(\sigma)$ is analytic on $|\sigma| < \beta^{\alpha}_{n+m+1}$, we have by the maximum modulus theorem $|w_{n+m+1}(\eta)/\eta^3| \leqslant k\beta^{3\alpha-1/2}_{n+m+1}/\beta^{3\alpha}_{n+m+1}$, so that

$$|w_{n+m+1}(\eta)| \leqslant 27kL^{-(3/2)(d-1/3)}\beta^{3\alpha-1/2}_{n+m+1}$$

These estimates, together with $\beta_{n+m+2} < 2L^{(d-2)}\beta_{n+m+1}$, easily imply

$$|\tilde{V}^{(n+m+1)}(r)| \leqslant 60kL^{-d-1/2}\beta^{(3\alpha-1/2)}_{n+m+2}$$

and

$$|e^{-L^d\tilde{V}^{(n+m+1)}(r)} - 1| \leqslant 70kL^{-1/2}\beta^{3\alpha-1/2}_{n+m+2}$$

Let

$$dv = \frac{1}{I} \exp\left[-\frac{1}{2}(1 + 8\lambda_{n+m+1} L^d) u^2 - \frac{1}{2} t^2 \right] \chi_o \, du \, dt$$

where I is a normalization such that $\int dv = 1$, and it is easy to show that I is bounded below by a strictly positive L-dependent constant. Write

$$\frac{N_o}{I} = \int \left(\prod_{l=n+m}^{n} M_l \right) \partial_2^2 V^{(n)}(r) \, dv + \int \left(\prod_{l=n+m}^{n} M_l \right) \partial_2^2 V^{(n)}(r) \times (e^{-L^d V^{(n+m+1)}(r)} - 1) \, dv \tag{3.6}$$

From the induction hypothesis,

$$\left(\prod_{l=n+m}^{n} M_l \right) \partial_2^2 V^{(n)}(r) = f_m^{(n)}(|r|) + \frac{t_2^2}{(\eta + \beta_{n+m+1}^{1/2})^2} g_m^{(n)}(|r|)$$

Also from the hypothesis, it follows that $|f_m^{(n)}(|r|)|$, $|h_m^{(n)}(|r|) - 8\lambda_n| \leqslant 3k\beta_n^{3\alpha - 1/2}$, which implies

$$\left| \int \left(\prod_{l=n+m}^{n} M_l \right) \partial_2^2 V^{(n)}(r)(e^{-L^d V^{(n+m+1)}(r)} - 1) \, dv \right| \leqslant 300k^2 L^{-1/2} \beta_n^{6\alpha - 1}$$

The first term in (3.6) is written as

$$\frac{N_o}{I} = \gamma_m^{(n)} + \int \bar{f}_m^{(n)}(\eta) \, dv + \int \frac{t_2^2 g_m^{(n)}(|r|)}{(\eta + \beta_{n+m+1}^{1/2})^2} \, dv \tag{3.7}$$

Since $\bar{f}_m^{(n)}(0) = 0$, we have by the maximum modulus theorem that

$$|\bar{f}_m^{(n)}(\eta)| \leqslant 6kL^{-(d-1/3)/2} \beta_{n+m+1}^{3\alpha - 1/2} \leqslant 12kL^{-5/6} \beta_{n+m+2}^{3\alpha - 1/2}$$

and the third integral in (3.7) is bounded by $100\lambda L^{-(d-1/3)/2} \beta_{n+m+1}^{2\alpha - 1}$. In conclusion we can write $N_o/I = \gamma_m^{(n)} + b_1(\sigma')$, where $b_1(\sigma')$ is analytic on $|\sigma'| < \frac{1}{2}(2L^{(d-2)} \beta_{n+m+1})^\alpha$ and $|b_1(\sigma')| \leqslant 15kL^{-5/6} \beta_{n+m+2}^{3\alpha - 1/2}$; the term D_o/I can be analyzed in the same way. In this way, we arrive at

$$f_{m+1}^{(n)}(|\phi|) = \frac{N_o/I + N_c/I}{D_o/I + D_c/I} = \gamma_m^{(n)} + b_2(\sigma')$$

with b_2 analytic on $|\sigma'| < \frac{1}{2}(2L^{(d-2)} \beta_{n+m+1})^\alpha$ and $|b_2(\sigma')| \leqslant 35kL^{-5/6} \beta_{n+m+2}^{3\alpha - 1/2}$. Now, letting

$$\Delta\beta_{n+m+1} = \beta_{n+m+2}^{1/2} - L^{(d-2)/2} \beta_{n+m+1}^{1/2} = O(\beta_{n+m+1}^{1/2})$$

we write $\sigma' = |\phi| - \beta_{n+m+2}^{1/2} + \Delta\beta_{n+m+1} = \sigma + \Delta\beta_{n+m+1}$, with $\sigma = |\phi| - \beta_{n+m+1}^{1/2}$. Notice that if the original β is large enough, the region $|\sigma| < \frac{1}{2}\beta_{n+m+2}^{\alpha}$ is contained in $|\sigma'| < \frac{1}{2}(2L^{(d-2)}\beta_{n.m.1})^{\alpha}$. Writing

$$f_{m+1}^{(n)}(|\phi|) = \gamma_m^{(n)} + b_2(\Delta\beta_{n+m+1}) + b_2(\sigma') - b_2(\Delta\beta_{n+m+1})$$

we define $\gamma_{m+1}^{(n)} = \gamma_m^{(n)} + b_2(\Delta\beta_{n+m+1})$, and $\tilde{f}_{m+1}^{(n)}(\sigma) = b_2(\sigma') - b_2(\Delta\beta_{n+m+1})$. Then, $\tilde{f}_{m+1}^{(n)}(\sigma)$ is analytic on $|\sigma| < \frac{1}{2}\beta_{n+m+2}^{1/2}$ and $|\gamma_{m+1}^{(n)} - \gamma_m^{(n)}| \leqslant 35kL^{-5/6}\beta_{n+m+2}^{3\alpha-1/2}$ and $|\tilde{f}_{m+1}^{(n)}(\sigma)| \leqslant 70kL^{-5/6}\beta_{n+m+2}^{3\alpha-1/2}$. Thus, since L is large, we see that the induction hypothesis for $f_m^{(n)}$ holds for $m+1$ if it holds for $m \geqslant -1$. The verification for $h_m^{(n)}$ is very similar and so we omit it. ∎

4. DECAY OF TRUNCATED TWO-POINT FUNCTION

In this section we obtain the decay rate of the perpendicular and parallel truncated two-point functions, thus proving Theorem 2. From Section 3, Theorem 8, we have the representation

$$\begin{aligned}\beta\langle\phi_i(x)\,\phi_i(y)\rangle &= \beta\langle\phi_i^2(0)\rangle^2\,\delta_{i1} + \frac{L^{-(n-1)(d-2)}}{1-L^{-(d-2)}} - (1-L^{-d})\,\delta_{i1}\cdot 8\\ &\quad\times\sum_{j=0}^{\infty}\lambda_{n+j}a_{n+j}^2 + L^{-d}8\delta_{i1}\lambda_{n-1}a_{n-1}^2 + E_n\end{aligned}$$

where

$$E_n = -(1-L^{-d})\sum_{j=0}^{\infty} a_{n+j}^2[O(\beta_{n-1}^{3\alpha-1/2})] + L^{-d}a_{n-1}^2\,O(\beta_{n-1}^{3\alpha-1/2})$$

From I, $\frac{1}{2}L^{n(d-2)}\beta \leqslant \beta_n \leqslant 3L^{n(d-2)}\beta$, and since

$$a_m = L^{-(m-2)(d-2)/2}\left(\frac{1-L^{-2m}}{1-L^{-2}}\right) \sim L^{-(m/2)(d-2)}$$

we see that

$$|E_n| < cL^{-n[d-2+(d-2)\delta]}\beta^{3\alpha-1/2} \tag{4.1}$$

where $\delta = 1/2 - 3\alpha > 0$. Thus, for the perpendicular two-point function $=$ $(2, 3,..., \nu)$ we see that

$$\lim_{n\to\infty} L^{(n-1)(d-2)}\langle\phi_i(x)\,\phi_i(y)\rangle = \frac{1}{\beta}\frac{1}{1-L^{-(d-2)}}$$

which is the exact decay of the hierarchical inverse Laplacian.

For the parallel case ($i=1$) we have

$$\begin{aligned}
&\beta(\langle \phi_1(x)\,\phi_1(y)\rangle - \langle \phi_1(0)\rangle^2)\\
&\quad = \frac{L^{-(n-1)(d-2)}}{1-L^{-(d-2)}} - (1-L^{-d})\cdot 8 \sum_{j=0}^{\infty} \lambda_{n+j} a_{n+j}^2 + L^{-d}\cdot 8\lambda_{n-1} a_{n-1}^2\\
&\qquad - (1-L^{-d}) \sum_{j=0}^{\infty} a_{n+j}^2 \cdot O(\beta_{n+j}^{3\alpha-1/2}) + L^{-d} a_{n-1}^2 \cdot O(\beta_{n-1}^{3\alpha-1/2})\\
&\quad \equiv S_0(n) + S_{1n}(\{\lambda_m\}) + S_2(\lambda_{n-1}) + E_n
\end{aligned} \tag{4.2}$$

where we define

$$S_{1n}(\{k_m\}) \equiv -(1-L^{-d})\cdot 8 \sum_{j=0}^{\infty} k_{n+j} a_{n+j}^2$$

and

$$S_2(k_{n-1}) \equiv L^{-d}\cdot 8k_{n-1} a_{n-1}^2$$

From I we have $\lim_{n\to\infty} \lambda_n = \lambda^* = (L^2-1)/8L^d$; thus,

$$\begin{aligned}
&\lim_{n\to\infty} L^{(n-1)(d-2)} \beta[\langle \phi_1(x)\,\phi_1(y)\rangle - \langle \phi_1(0)\rangle^2]\\
&\qquad = \frac{1}{1-L^{-(d-2)}} - 8(1-L^{-d}) \sum_{j=0}^{\infty} \lambda^* L^{-(j-1)(d-2)}\\
&\qquad\quad \times \frac{1}{(1-L^{-2})^2} + \frac{8L^{-d}\lambda^* L^{(d-2)}}{(1-L^{-2})^2} = 0
\end{aligned} \tag{4.3}$$

showing that the truncated two-point function in the parallel direction decays faster than the perpendicular one.

We now obtain faster decay than that given by (4.3). Note that the sum of the first three terms of (4.2), with λ^* replacing λ_m, is given by

$$\begin{aligned}
&S_0 + S_{1n}(\{\lambda^*\}) + S_2(\lambda^*)\\
&\quad = \frac{L^{-(n-1)(d-2)}}{1-L^{-(d-2)}} - (1-L^{-d})\, 8\lambda^* \sum_{j=0}^{\infty} a_{n+j}^2 + L^{-d} 8\lambda^* a_{n-1}^2\\
&\quad = \frac{L^d}{1-L^{-(d-2)}} L^{-n(d+2)}
\end{aligned} \tag{4.4}$$

i.e., the $L^{-n(d-2)}$ and L^{-nd} terms cancel. We can obtain faster falloff by estimating the rate of convergence of λ_n to λ^*. We have, letting $K_n \equiv \lambda_n^{-1} - \lambda^{*-1}$, $\lambda_0 = \lambda$, and $R_n = L^{2n} K_n - K_0$, the following result.

Lemma 4.1. For λ and β large, we have:

(a) $|\lambda-\lambda_n|\leqslant c\beta^{\alpha-1/2}L^{n(\alpha-1/2)}$, for $d\geqslant 3$.

(b) For d large

$$\lambda^*-\lambda_n=L^{-2n}\lambda^2(K_0+R_\infty)-L^{-4n}\lambda^3(K_0+R_\infty)^2+O(L^{-(4+\varepsilon)n})$$

where $\varepsilon>0$, $R_\infty\equiv\lim_{n\to\infty}R_n$, and $|R_\infty|<c\beta^{\alpha-1/2}$.

Proof. From I we have

$$K_n=\lambda_n^{-1}-\lambda^{*-1}=(\lambda^*-\lambda_n)(\lambda_n\lambda^*)^{-1}=L^{-2n}K_0+L^{-2n}R_n$$

where

$$R_n=c(L)\sum_{j=0}^{n-1}L^{2j}O(\beta_j^{\alpha-1/2})$$

(a) As $\beta_j\sim L^{j(d-2)}\beta$ implies $L^{-2n}|R_n|\leqslant c\beta^{\alpha-1/2}L^{n(d-2)(\alpha-1/2)}$, the result follows.

(b) For large d, note that $|R_n|<c\beta^{\alpha-1/2}$ and $|R_n-R_\infty|<cL^{-(2+\varepsilon)n}$ with $\varepsilon>0$. Now, by writing $\lambda^*-\lambda_n=L^{-2n}K_0\lambda_n\lambda^*+L^{-2n}R_n\lambda_n\lambda^*$, we see that $|\lambda^*-\lambda_n|\leqslant L^{-2n}c$.

Substituting $\lambda+(\lambda_n-\lambda)$ for λ_n on the right side and iterating, we arrive at

$$\begin{aligned}\lambda^*-\lambda_n&=L^{-2n}K_0\lambda^2-L^{-2n}R_n\lambda^{*2}+L^{-2n}(K_0+R_n)\,\lambda^*\\&\quad\times[-L^{-2n}K_0\lambda^{*2}-L^{-2n}R_n\lambda^{*2}-L^{2n}(K_0+R_n)(\lambda_n-\lambda^*)]\\&=L^{-2n}K_0\lambda^{*2}+L^{-2n}R_n\lambda_n^{*2}-L^{-4n}\lambda^{*3}(K_0+R_n)^2+O(L^{-6n})\end{aligned}$$

Now write $R_n=R_\infty+(R_n-R_\infty)$, so that the above becomes

$$\lambda^*-\lambda_n=L^{-2n}\lambda^{*2}(K_0+R_\infty)-L^{-4n}\lambda^{*3}(K_0+R_\infty)^2+O(L^{-(4+\varepsilon)n})$$

Now we estimate the falloff by writing $\lambda_m=\lambda^*+(\lambda_m-\lambda^*)$ in (4.2). Thus

$$\begin{aligned}\beta\langle\phi_1(x)\,\phi_1(y)\rangle^T&=S_0+S_{1n}(\{\lambda^*\})+S_2(\lambda^*)+S_{1n}(\{\lambda_m-\lambda^*\})\\&\quad+S_2(\lambda_{n-1}-\lambda^*)+E_n\end{aligned}\tag{4.5}$$

and for all $d\geqslant 3$, using Lemma 4.1(a),

$$|S_{1n}(\{\lambda_m-\lambda\})|,\ |S_2(\lambda_{n-1}-\lambda)|\leqslant c\beta^{\alpha-1/2}L^{-n[d-2+\delta_1]}$$

with $\delta_1 = 1/2 - \alpha > 0$. Combining these estimates with (4.1) and (4.4) gives

$$\beta\langle\phi_1(x)\,\phi_1(y)\rangle^T \leqslant cL^{-n[d-2+\delta]}, \qquad d \geqslant 3$$

Now we consider d large. Using Lemma 4.1(b) in (4.5), we get

$$\begin{aligned}\beta\langle\phi_1(x)\,\phi_1(y)\rangle^T = {} & S_0 + S_{1n}(\{\lambda\}) + S_2(\lambda) - \lambda^2(K_0 + R_\infty) \\ & \times [S_{1n}(\{L^{-2m}\}) + S_2(L^{-2(n-1)})] \\ & + \lambda^{*3}(K_0 + R_\infty)^2\,[S_{1n}(\{L^{-4m}\}) + S_2(L^{-4(n-1)})] \\ & + S_{1n}(O(L^{-n(4+\varepsilon)})) + S_2(O(L^{-(n-1)(4+\varepsilon)})) + E_n \end{aligned} \tag{4.6}$$

Now

$$S_{1n}(\{L^{-2m}\}) + S_2(L^{-2(n-1)}) = \frac{-2L^d L^{-n(d+2)}}{\lambda(1 - L^{-(d+2)})} + O(L^{-n(d+4)}) \tag{4.7}$$

$$S_{1n}(\{L^{-4m}\}) + S(L_2^{-4(n-1)}) = \frac{L^d L^{-n(d+2)}}{\lambda(1 - L^{-(d+2)})} + O(L^{-n(d+4)}) \tag{4.8}$$

and the last three terms in (4.6) are bounded by $cL^{-n(d+2+\varepsilon)}$. Using (4.1), (4.4), (4.7), and (4.8) in (4.6), we obtain, noting that $1 + \lambda^* K_0 = \lambda^*/\lambda_0$,

$$\begin{aligned}\langle\phi_1(x)\,\phi_1(y)\rangle^T & \\ = {} & \frac{1}{\beta}\frac{L^d L^{-n(d+2)}}{1 - L^{-(d+2)}}[1 + 2\lambda^*(K_0 + R_\infty) + \lambda^{*2}(K_0 + R_\infty)^2] \\ & + O(L^{-n(d+2+\varepsilon)}) \\ = {} & \frac{1}{\beta}\frac{L^d L^{-n(d+2)}}{1 - L^{-(d+2)}}\lambda^{*2}\left(\frac{1}{\lambda} + R_\infty\right)^2 + O(L^{-n(d+2+\varepsilon)})\end{aligned}$$

5. CONCLUDING REMARKS

Here we have shown that the decay of the parallel two-point function is faster than the perpendicular one in all $d \geqslant 3$ and $|x-y|^{-(d+2)}$ decay for large d. We expect $|x-y|^{-(d+2)}$ decay for all $d \geqslant 3$ based on the results of I that the shifted action has the noncanonical Gaussian fixed point $4\lambda^*\phi_1^2$ and $|x-y|^{-(d+2)}$ is the decay associated with this fixed point. A calculation showing the finiteness of the zero-field susceptibility would indicate at least $|x-y|^{-(d+\varepsilon)}$, $\varepsilon > 0$, falloff of the two-point function. One can show finiteness for $d > 6$, but it would take higher-order perturbation calculations to reach $d \geqslant 3$. In addition to the question of decay it would be interesting to know the behavior of the model in the critical region.

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